IUCAA Refresher Course: Gravitational Waves: Foundations

References

Short reviews:

1. Eanna Flannagan and Scott Hughes, Basics of gravitational wave theory. gr-qc/0501041

2. Enrico Barausse, Physics of gravitational waves. 2303.11713

3. Jose P. S. Lemos, When spacetime vibrates: an introduction to gravitational waves. 2512.22679

Lecture notes:

1. General Relativity by Professor Scott Hughes. See Lectures 14, 15, 16, 17. Lectures 14-17

2. General Relativity by Professor David Tong. See the chapter 'When gravity is weak'. lecture-notes.

Dedicated books on GW:

1. Jolien D. Creighton and Warren Anderson, Gravitational-wave physics and astronomy.

2. Michele Maggiore, Gravitational Waves: Volume 1: Theory and Experiments.

Some useful GR cum GW texts:

1. C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation.

2. L. D. Landau and E. M. Lifshitz, Classical theory of fields.

3. B. F. Schutz, A first course in general relativity.

4. T. Moore, A general relativity workbook

5. S. V. Dhurandhar and S. Mitra, General relativity and gravitational waves (last two chapters).

6. A. P. Lightman, W. H. Press, R. H. Price, S. A. Teukolsky, Problem book in relativity and gravitation.

Tutorial 1

* *marked problems are not directly linked to the lectures, but are important.*

1. (a) Show, through a step-by-step calculation, how Newton's second law for a potential Φ is recovered by analysing the geodesic equation in a line element where $g_{ij} = \eta_{ij} + h_{ij}$ (h_{ij} small). Note the identification of h_{00} with $\frac{2\Phi}{c^2}$.

(b) Show how Poisson's equation for gravity $\nabla^2\Phi = 4\pi G\rho$ follows from the linearised Einstein field equations. In what sense (not the obvious difference in the names of quantities!) do you think $\nabla^2\Phi = 4\pi G\rho$ in *gravito-statics* is different from the formally similar Poisson's equation $\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$ in electrostatics.

2. Write down the geodesic deviation equation for the deviation vector η^i in a spacetime $g_{ij} = \eta_{ij} + h_{ij}$. You need to retain only linear in h terms.

3. (a) Write down the Killing equations for the metric $g_{ij} = \eta_{ij} + h_{ij}$, where h_{ij} are functions of t and z only. Find some of the Killing vectors.

(b) Show that the Riemann tensor for the metric $g_{ij} = \eta_{ij} + h_{ij}$, at linear order in h_{ij}, is invariant under the gauge transformation (i.e. $x'^i = x^i + \xi^i, h'_{ij} = h_{ij} - \partial_i\xi_j - \partial_j\xi_i$).

4. (a) Find the angular velocity of a zero-angular-momentum observer (ZAMO) in Kerr spacetime.

(b) Consider the Killing vectors $t^i \equiv (1, 0, 0, 0)$ and $\phi^i \equiv (0, 0, 0, 1)$ in Kerr spacetime. Find the equations of the surfaces where the norm of t^i and $t^i + \Omega\phi^i$ become null. Identify them as the static limit surfaces and the stationary limit surfaces in Kerr.

(c) The angular momentum and mass of the Sun and Earth are:

$$J_\odot = 1.92 \times 10^{41}\, kg\, m^2\, s^{-1} \;;\; M_\odot = 1.98 \times 10^{30}\, kg$$
$$J_\oplus = 7.10 \times 10^{33}\, kg\, m^2\, s^{-1} \;;\; M_\oplus = 5.98 \times 10^{24}\, kg$$

Find a and M (Kerr metric parameters) in 'length' units. If the Sun and the Earth collapse to respective Kerr metrics without shedding angular momentum, will they form a black hole or a naked singularity? Explain your answer.

* **5.** Consider the following Lagrangian density (Fierz-Pauli):

$$\mathcal{L} = -\frac{1}{4}\partial_k h^{ij}\partial^k h_{ij} + \frac{1}{2}\partial_k h_{ij}\partial^j h^{ik} + \frac{1}{4}\partial_k h \partial^k h - \frac{1}{2}\partial_j h^{ij}\partial_i h$$

(a) Show that on varying the above action w.r.t. h_{ij} one does obtain the linearised Einstein equations in vacuum.

(b) Use the Lorentz gauge condition $\partial_i \bar{h}^{ij} = 0$, to obtain the gauge-fixed action.

(c) Do a variation w.r.t. $\bar{h}_{ij}$. Find the corresponding equation of motion.

(d) What could be the additional matter term in the action?

(e) Using the gauge-fixed Lagrangian density find the energy momentum tensor using its canonical definition.

Tutorial 2

** marked problems are not directly linked to the lectures, but are important.*

1. Consider a plane wave metric perturbation $h_{ij} = A_{ij} \sin(k_m x^m)$, where A_{ij} is a constant matrix and k^i is the wave number four-vector.

(a) Find the Riemann and Ricci tensor components for this h_{ij}.

(b) Show that, when $k^2 \neq 0$, the metric perturbation will satisfy the vacuum Einstein equations $R_{ij} = 0$ only if $A_{ij} = \frac{1}{k^2}(k_i A_j + k_j A_i)$, where $A_i = k_j \eta^{jk} A_{ki} - \frac{1}{2} k_i A$, with A being the trace of A_{ij}.

(c) Show that for $k^2 \neq 0$, the condition in (b) above for satisfying $R_{ij} = 0$, also leads to $R_{ijkl} = 0$.

(d) Show that with $k^2 = 0$, the metric perturbation can represent a gravitational wave. Check if such a wave does move at the speed of light.

2. For a plane gravitational wave propagating along z the Riemann tensor components R_{ijkl} are functions of $t - \frac{z}{c}$. Show, using the symmetry properties of the Riemann tensor components and the Bianchi identity, that the only independent components of R_{ijkl} are $R_{0\alpha 0\beta} = \mathcal{E}_{\alpha\beta}$. Analyse the physical effects (six different polarisations) of the tidal tensor $\mathcal{E}_{\alpha\beta}$ on a sphere of particles by using (via geodesic deviation) the expression for the tidal acceleration a_α.

3. Consider a gravitational wave given as $\bar{h}^{ij} = A^{ij} \cos(k^m x_m)$ where

$$A^{ij} = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & c & b & 0 \\ -a & b & -c & a \\ 0 & 0 & a & 0 \end{pmatrix}$$

Does this wave satisfy the Lorenz gauge condition? What is the direction of propagation? Using the algebraic projection method discussed in the lecture, find a ξ^i which takes this metric perturbation into a transverse, traceless (TT) gauge. Write the A^{ij} in the TT gauge.

4. You are given the following gravitational wave

$$h_{ij}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A\cos\left(k_m x^m\right) & A\sin\left(k_m x^m\right) & 0 \\ 0 & A\sin\left(k_m x^m\right) & -A\cos\left(k_m x^m\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $k_t = -\omega$, $k_x = k_y = 0$, $k_z = \omega$ and A is a suitably small constant.

(a) Show that the above is a superposition of a plus polarised wave and a cross-polarised wave that is 90 degrees out of phase. It is thus, a circularly polarised wave.

(b) Argue that such a wave perturbs a ring of particles in the xy plane in such a way that their shape becomes an ellipse, which rotates in that plane. What is rotation rate of the ellipse in terms of ω?

5. Assume a plane gravitational wave propagating along the z direction. We intend to study the effect of the propagating GW on relative motion of two nearby particles. Let the particles be separated by the vector $\vec{\zeta} = \zeta\left(\sin\theta\cos\phi\,\mathbf{e_1} + \sin\theta\sin\phi\,\mathbf{e_2} + \cos\theta\,\mathbf{e_3}\right)$.

(a) Write down the relative acceleration a_α using the geodesic deviation equation.

(b) Take $\theta = \frac{\pi}{2}$, i.e. particles in the plane transverse to the direction of propagation of the GW. If $(x, y) = (\zeta\cos\phi, \zeta\sin\phi)$, then find the vector field (a_1, a_2) when $h_+ \neq 0, h_\times = 0$ and when $h_+ = 0, h_\times \neq 0$. Plot these vector fields (lines of force) and note that the names 'plus' and 'cross' originate from the nature of the vector fields in these plots.

6. Consider a purely plus polarised gravitational wave in the TT gauge, traveling along the z direction and given by the line element:

$$ds^2 = -dt^2 + \left(1 + h_+(t,z)\right)dx^2 + \left(1 - h_+(t,z)\right)dy^2 + dz^2$$

(a) Obtain the Ricci tensor components upto second order in the perturbation h_+.

(b) What is the time averaged (over many cycles) effective energy density (assuming h_+ is a harmonic function of $t - z$) of this gravitational wave?

(c) Generalise to include $h_\times$ in the line element and repeat the calculations done in (a) and (b) above.

*** 7.** Consider two masses m_1 and m_2 placed on the x axis and connected by a spring of spring constant k. The equilibrium distance between the masses is L. x measures the displacement of the masses w.r.t. equilibrium. A plane gravitational wave traveling along z and polarised along x, such that $h_{xx} = -h_{yy} = h_+ = h \cos\omega t$ is incident on the oscillator.

(a) Write down the equation for driven oscillations by including a dissipative force $F_d = -b\dot{x}$ ($b = 2\mu\beta, \mu$ is reduced mass).

(b) Assuming resonance occurs at $\omega \sim \omega_0$ (natural frequency of oscillator) what is the $x_{max,res}$?

(c) Find $\langle E_{kin}\rangle$, $\langle E_{pot}\rangle$ and $\langle W\rangle = -\langle \frac{dE_{tot}}{dt}\rangle$ where $\langle ..\rangle$ denotes averaging over a cycle of oscillation.

(d) Suppose a 1 kHz gravitational wave with $h \sim 10^{-21}$ is incident, for a long duration, on such a system ($L \sim 1$ m, $\mu = 1000$ kg and quality factor $Q \sim \frac{\omega_0}{2\beta} = 10^6$), find $x_{max,res}$. Find $\langle E_{tot}\rangle$ and the thermal energy at $T = 300$ K. Can we distinguish oscillations due to GW from thermal oscillations?

*** 8.** Find the equation which $H(x, y, u)$ must satisfy if the following line element has to be a vacuum solution of the Einstein equations.

$$ds^2 = -H(u, x, y)du^2 - 2du dv + dx^2 + dy^2$$

What is a possible form for $H(u, x, y)$? This is known as the *pp-wave solution.*

Tutorial 3

** marked problems are not directly linked to the lectures, but are important.*

1. Obtain the antenna pattern functions F_+ and $F_\times$ (ϕ, θ, ψ are the Euler angles) as given below.

$$\begin{aligned} F_+ &= \frac{1}{2}\left(1+\cos^2\theta\right)\cos 2\phi\cos 2\psi - \cos\theta\sin 2\phi\sin 2\psi \\ F_\times &= -\frac{1}{2}\left(1+\cos^2\theta\right)\cos 2\phi\sin 2\psi - \cos\theta\sin 2\phi\cos 2\psi \end{aligned}$$

2. Show that for the standard unit normal (n^α) along the radial direction, we have,

$$\int n_\alpha n_\beta \; d\Omega = \frac{4\pi}{3}\delta_{\alpha\beta};$$

$$\int n_\alpha n_\beta n_\gamma n_\delta \; d\Omega = \frac{4\pi}{15}\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$$

3. Compute the GW amplitudes for an observer in an arbitrary direction at an angle ι with the orbital spin axis of the binary system. Assume that the orbit of the binary (masses $\frac{M}{2}$ each) to be circular (radius R). Obtain the following expressions for the GW signal in the TT gauge.

$$\begin{aligned} h_+ &= -\left(\frac{GM}{c^2 r}\right)\left(\frac{GM}{c^2 R}\right)\left(\frac{1+\cos^2\iota}{2}\right)\cos 2\omega t \\ h_\times &= -\left(\frac{GM}{c^2 r}\right)\left(\frac{GM}{c^2 R}\right)(\cos\,\iota)\sin 2\omega t \end{aligned}$$

4. Consider two point masses $m_1 \neq m_2$ and total mass M, in circular orbits at a separation distance a and an orbital frequency ω.

(a) Compute the quadrupole tensor $I_{\alpha\beta}$. Choose the orbit in the xy plane. Let $a_1 = \frac{m_2}{M}a$, $a_2 = \frac{m_1}{M}a$ and $a = a_1 + a_2$ with the centre of mass at the origin. Show that, $I_{xx} = \frac{1}{2}\mu a^2\cos\,2\phi(t) = -I_{yy}$, $I_{xy} = \frac{1}{2}\mu a^2\sin\,2\phi(t) = I_{yx}$, $I_{xz} = I_{zx} = I_{zy} = I_{yz} = I_{zz} = 0$. Here $\phi(t) = \int\omega(t)dt$ and $\omega(t)$ adiabatically evolves with time. Also $\mu = \frac{m_1 m_2}{M}$ is the reduced mass.

(b) Discuss the case when the masses are equal.

*** 5.** A gravitational wave encounters a viscous fluid, which is initially at rest, with fluid four-velocity given as $u^i = (1, 0, 0, 0)$.

(a) The shearing of the fluid is described by the shear tensor:

$$\sigma_{ij} = \frac{1}{2}\left(\nabla_i u_j + \nabla_j u_i + u_i u^k \nabla_k u_j + u_j u^k \nabla_k u_i\right) - \frac{1}{3}(g_{ij} + u_i u_j)\nabla_k u^k$$

Show that the shear caused by the gravitational wave in the TT gauge is purely spatial with

$$\sigma_{\alpha\beta} = \frac{1}{2}\frac{\partial}{\partial t} h^{TT}_{\alpha\beta}$$

(b) The shearing of the fluid generates a contribution to the stress energy tensor of the form $T^{viscosity}_{ij} = -2\eta\sigma_{ij}$, where η is the coefficient of viscosity. The linearised field equations for the gravitational wave are therefore:

$$\Box h^{TT}_{ij} = \eta \frac{16\pi G}{c^4}\frac{\partial}{\partial t} h^{TT}_{ij}$$

Show that a plane wave traveling along the z-axis is attenuated by the fluid by an amount $e^{-\frac{z}{\ell}}$ where ℓ is the attenuation length scale given as

$$\ell = \frac{c^3}{8\pi G \eta}$$

(c) Ketchup has a coefficient of viscosity $\eta = 50$ kg m^{-1} s^{-1}. Calculate the distance a gravitational wave must travel through ketchup, before it is attenuated by a factor $\frac{1}{e}$.

Tutorial 4

1. (a) Show that the total GW luminosity for the binary in Tutorial 3, Problem 4 (a) is:

$$L_{GW} = \frac{32}{5}\frac{G\mu^2 a^4 \omega^6}{c^5} = \frac{32}{5}\frac{G^4 M^3 \mu^2}{c^5 a^5}$$

(b) Consider an equal mass binary system with each component having mass M kg. The binary has a decaying circular orbit and emits gravitational radiation. Show that the luminosity (in ergs/s) of the emitted gravitational radiation, assuming the orbital period given as P (in hours) can be written as (show all calculations):

$$L_{(GW)} \sim 1.9 \times 10^{33} \left(\frac{M}{M_\odot}\frac{1\ hour}{P}\right)^{\frac{10}{3}}$$

2. Calculate the gravitational radiation luminosity of a spinning thin metal rod of mass M and length L , spinning at an angular frequency ω about the symmetrical, say z axis passing through its centre. If the mass of the rod is 10 kg, length 1 m and the frequency of rotation $\nu = 1\, kHz$, estimate the value of the GW luminosity.

3. The first detected binary black hole merger, GW150914, with component masses 30 and 35 solar masses has the maximum luminosity of $200 M_\odot\, c^2\, s^{-1} \equiv 3.6 \times 10^{56} erg\, s^{-1}$ in the frequency range 100-200 Hz. Check the following using the results obtained in the lectures (or use your results in Tutorial 4, Problem 1 above). :

(a) Using $n = 3.5$ (recall that $2R = nm$ with $m = \frac{GM}{c^2}$) find an estimate for the frequency of the emitted GW.

(b) For $n = 3.5$ estimate the peak luminosity (You have to use the luminosity formula involving M, n, m and c).

(c) GW150914 is about 400 Mpc away. How much (find the factor) is the flux reduced when it reaches us?

4. GW190521 is an observed merger event of two black holes with masses $85M_\odot$ and $66M_\odot$. The peak GW frequency observed is around 60 Hz.

(a) Find the peak luminosity of GW190521 in terms of $M_\odot c^2$. Is it close to the estimated value of $208 M_\odot c^2$?

(b) Using the Peters-Mathews eccentricity factor $f(e) = \frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{\frac{7}{2}}}$ can you approximately estimate the value of the eccentricity for which the calculated peak luminosity in (a) may match with the observed value of $208M_{\odot}c^2$?

5. Consider the radial infall of a particle of mass m towards a black hole of mass M ($m << M$). The particle comes in from infinity (zero velocity) to a finite value of z.

(a) Obtain expressions for $\dot{z}(t)$, $\ddot{z}(t)$ and $\dddot{z}(t)$ in terms of $z(t)$ using standard Newtonian mechanics.

(b) Find the second mass moment I_{ij} of the falling particle of mass m. Also write down its traceless piece $\mathcal{I}_{ij}$.

(c) Obtain the expression $\frac{dE}{dt}$ (energy in emitted gravitational radiation per unit time) without any time averaging.

(d) Show that the total radiated energy (integrated from $z = \infty$ to $z = R$), as the particle falls in from infinity to say $z = R$, is given as ($r_s = \frac{2GM}{c^2}$):

$$E = \frac{4}{105}\frac{Gm^2}{r_s}\left(\frac{r_s}{R}\right)^{\frac{7}{2}}$$

(e) Show that the total energy radiated if the particle reaches $R = r_s$ is $0.019mc^2\left(\frac{m}{M}\right)$. *Note: this is quite close to the fully relativistic calculation for which the overall numerical factor is 0.010.*

Tutorial 5

1. (a) Consider a binary system with equal masses, $10M_\odot$ each. It is located at a distance 200 Mpc. Using Kepler's third law and $f_0 = 10$ Hz find a_0. From the fact that $a(t = t_c) = 0$ find t_c. Write down the explicit expressions (with all numerical factors) for $a(t)$, $\omega(t)$ and $\phi(t)$. Thereafter, write down the full expression for h_+ assuming a zero inclination angle ι. Plot h_+ near $t = t_c$ and demonstrate its chirp form.

(b) As a generalisation, consider again the case of two point masses $m_1 \neq m_2$ and total mass M in circular orbits at a separation distance a and frequency Ω. Use the expressions for L_{GW} and the energy in the orbit to obtain the following:

(i) the differential equation for the decaying separation $a(t)$.

(ii) the coalescence time t_c.

(iii) Define the chirp mass $\mathcal{M} = \left[\mu M^{\frac{2}{3}}\right]^{\frac{3}{5}}$. Obtain the expressions for the GW strain amplitude $\mathcal{A}(t)$, phase $\phi(t)$.

2. Consider an equal mass binary system with total mass M kg. The binary has a decaying circular orbit and emits gravitational radiation.

(a) Obtain an expression for the chirp mass $\mathcal{M}$ in terms of f_{GW} ($\omega = \pi f_{GW}$) and its time derivative $\dot{f}_{GW}$.

(b) Integrate to obtain $f_{GW}(t)$ as a function of t. What will you plot on the x and y axis so that the slope of the straight line graph can be used to find the chirp mass $\mathcal{M}$?

3. Consider the *cumulative periastron shift in time* observed by Hulse and Taylor over an extended period, for the binary pulsar PSR1913+16. The graph of observations is shown below.

To understand how one may get the values on the graph, do the following:

(a) Assume the first measured orbit to be around 7.75 hours measured by locating the periastron. If the value of $\frac{d\tau}{dt} = -2.422 \times 10^{-12}$, then by how much time is the next orbit shorter?

(b) Further, by how much time is the following orbit shorter?

(c) In 10 years, how many orbits are completed?

(d) Using the result in (c), calculate the cumulative shift (sum the series with no of terms

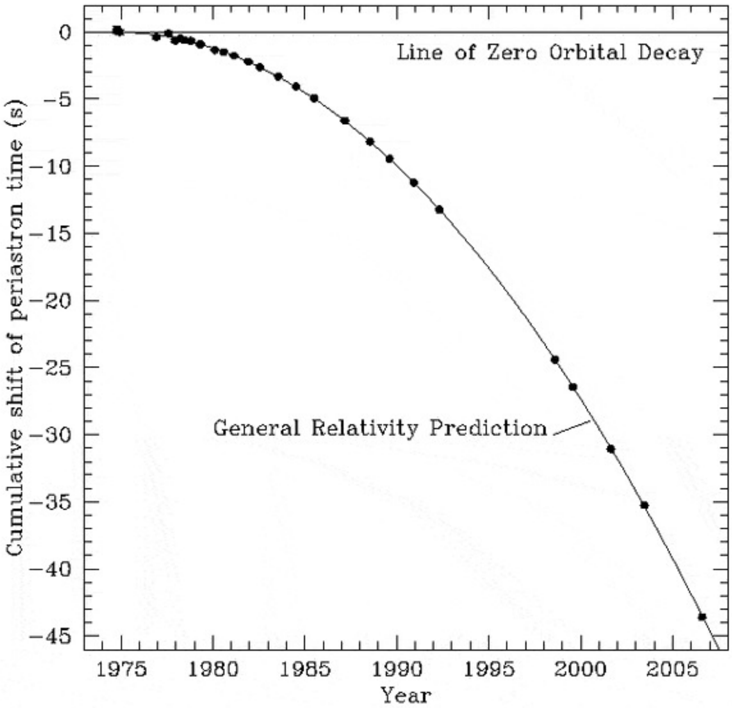

FIG. 1: Figure for Problem 7

= no. of orbits) and show that the value matches roughly with what you see on the graph.
(e) Using the Newtonian equation for $a(t)$ with the Peters-Mathews eccentricity correction factor, obtain the time of coalescence of the pulsar and its companian.
(f) Estimate the approximate magnitude of the gravitational wave strain which will have to be measured in a laser interferometric detector if it receives a signal from the binary PSR B 1913 +16. The distance to PSR B 1913+16 is 20,870 Ly.

4. PSR J0737-3039 is a recently (2003) discovered binary pulsar system consisting of two neutron stars, one with a mass of 1.337 $M_{\odot}$ and the other with a mass of 1.250 $M_{\odot}$ (solar mass $M_{\odot} = 2 \times 10^{30}$ kg), orbiting with a period of 2.4 hrs. Their orbit is only mildly eccentric ($e = 0.088$), and we are viewing the orbit almost edge-on. This system is about 1800 Light years (Ly) away ($1Ly = 9.46 \times 10^{15} m$) and it is known to have a decaying orbital period due to GW emission. Find the following using the equal mass (take the average of the two masses as the mass of each object) and circular orbit approximations (for answering (a), (b), (c) and (d)).

(a) The gravitational wave amplitude A_+, as would be received on Earth.

(b) The total gravitational wave power this object emits in watts.

(c) The rate $\frac{dP}{dt}$ at which its period is changing and the cumulative reduction of the period in 10 years (assuming a circular orbit and equal average masses).

(d) Qualitatively plot the 'theoretical' cumulative periastron shift as a function of time beginning at a value of $t = t_0$.

5. (a) GW250114 is the latest observed signal from a black hole merger of two black holes of masses 33.6 $M_\odot$ and 32.2 $M_\odot$. Assume an emitted GW frequency of 200 Hz. The distance of the system from Earth is found to be approximately 1.14×10^9 Ly.

(i) In an equal mass, circular orbit approximation (take the average of the two masses as the value for each mass) find the value of the luminosity in terms of $M_\odot c^2$.

(ii) Find the magnitude of the GW strain h for this system and the chirp mass (without using equal mass approximation).

(b) Look up the list of gravitational wave events listed here and try to estimate, for different events, various quantities like chirp mass, peak luminosity, radiated energy etc.